Published by Institute of Physics Publishing for SISSA/ISAS
Received: *October 10, 2002*
Accepted: *October 10, 2002*

Strings in time-dependent orbifolds

Hong Liu and Gregory Moore

Department of Physics, Rutgers University
Piscataway, New Jersey, 08855-0849, USA
E-mail: liu@physics.rutgers.edu, gmoore@physics.rutgers.edu

Nathan Seiberg

School of Natural Sciences
Einstein Drive, Princeton, NJ 08540, USA, and
Institute for Advanced Study
Einstein Drive, Princeton, NJ 08540, USA
E-mail: seiberg@ias.edu

Abstract: We continue and extend our earlier investigation "Strings in a Time-Dependent Orbifold" (hep-th/0204168). We formulate conditions for an orbifold to be amenable to perturbative string analysis and classify the low dimensional orbifolds satisfying these conditions. We analyze the tree and torus amplitudes of some of these orbifolds. The tree amplitudes exhibit a new kind of infrared divergences which are a result of some ultraviolet effects. These UV enhanced IR divergences can be interpreted as due to back reaction of the geometry. We argue that for this reason the three dimensional parabolic orbifold is not amenable to perturbation theory. Similarly, the smooth four dimensional null-brane tensored with sufficiently few noncompact dimensions also appears problematic. However, when the number of noncompact dimensions is sufficiently large perturbation theory in these time dependent backgrounds seems consistent.

Keywords: Bosonic Strings, Superstrings and Heterotic Strings, Conformal Field Models in String Theory.

JHEP10(2002)031

Contents

1. Introduction

Time dependent physics poses a large number of interesting conceptual and technical problems in quantum field theory, quantum gravity, and in string theory. So far very little work has been done on time-dependent backgrounds in the framework of string theory, although the situation is beginning to change (see, e.g. [1]–[22] for recent discussions of this issue). Here, we continue the investigation of strings in time dependent orbifolds which we started in [15]. We are adopting a conservative approach, simply trying to follow the standard rules of perturbative string theory, and trying to imitate the constructions and methods which have been proven useful in time independent setups.

In order to be able to apply perturbative string methods we have to deal with the following issues:

1. We start with a solution of the classical equations of motion; i.e. a worldsheet conformal field theory. This conformal field theory is most tractable when it is based on an orbifold of flat spacetime. Therefore, we will mod out $\mathbb{R}^{1,n}$ by a subgroup Γ of its symmetry group which is the Poincare group in $n+1$ dimensions $\mathcal{P}(1,n)$. We will denote the orbifold by $\mathcal{O} = \mathbb{R}^{1,n}/\Gamma$. In order to have nontrivial time dependence Γ should not be a subgroup of the euclidean group in n dimensions as in ordinary euclidean orbifolds.

2. In euclidean orbifolds the singularities of the target space are at fixed points of Γ. Lorentzian orbifolds typically have also other potential problems. The quotient by Γ can make time non-orientable. It can also create closed null curves (CNC's) or closed time-like curves (CTC). These are potentially problematic because they can lead to divergent expectation values of composite operators like the stress tensor and to large back reaction of the metric. We examine orbifolds which are time-orientable and have no CTC's, although we will allow singularities, non-Hausdorff spaces, and the possibility of CNC's. We will also see examples in which the orbifold has no CNC's but has closed curves whose invariant length is arbitrarily small; we will examine the consequences of these curves.

3. In order to help ensure the stability of the background against radiative corrections we will look for orbifolds which leave unbroken some amount of supersymmetry. This supersymmetry can guarantee that some of the back reaction due to CNC when they are present is harmless. Having such supersymmetry leads to a null Killing vector ∂_{x^-} (see [3, 4] and section 7 below). We can therefore use lightcone frame and treat x^+ as time. The orbifold $\mathcal{O}$ is thus foliated by spaces $\mathcal{F}_{x^+}$ of fixed x^+. The lightcone description also guarantees that the vacuum of the second quantized theory is trivial, and there is no particle production in the system [23].

4. Even if supersymmetry guarantees that the zero point function and the one point functions vanish to all orders, the question of back reaction is far from obvious. Vertex operators correspond to small deformations of the background. If they are singular at some point in spacetime, they can lead to infinite energy density there and to large change in the gravitational field. Such an effect can render perturbation theory invalid. For smooth backgrounds vertex operators which are singular at some point in spacetime can be excluded, but in various singular orbifolds all vertex operators are singular. We will discuss this issue in more detail below in some examples. We will also show that sometimes a smooth background with smooth vertex operators can also suffer from large back reaction. In particular, we will see that the back reaction is reflected in "UV enhanced IR divergences", which may also signal that one needs to consider more subtle asymptotic states.

In section 2 we will examine two orbifolds. The first, which we will refer to as the parabolic orbifold (it is also called the "null orbifold"), was introduced in [1] and briefly studied in [24]. Its geometry was further explored recently in [11] and strings in this background were investigated in [15]. The second orbifold, the "null-brane" was described in [4, 11]. In sections 3–6 we will slightly extend the analysis of string theory in the background of the parabolic orbifold in [15], and will apply the same techniques to the case of the null-brane background. In section 3 we consider the functions on the orbifolds which are important when constructing vertex operators. In section 4 we consider the canonical quantization of free strings. In section 5 we analyze the torus partition function, and in section 6 we consider tree level S-matrix amplitudes. In section 7 we comment on the general classification of such models and describe several new orbifolds which exhibit new phenomena.

JHEP10(2002)031

Strings in the null-brane orbifold are also studied by D. Robbins and S. Sethi [25] and by M. Fabinger and J. McGreevy [26]. Also, A. Lawrence [21] and G. Horowitz and J. Polchinski [22] reached conclusions related to ours about the back reaction and the validity of perturbation theory in the parabolic orbifold and the null-brane.

2. The parabolic orbifold and the null-brane

In [15] we analyzed in detail string theory on a parabolic orbifold, with target space $(\mathbb{R}^{1,2}/\Gamma)\times\mathcal{C}^{\perp}$ where $\Gamma\cong\mathbb{Z}$ is a parabolic subgroup of the three dimensional Lorentz group $\mathrm{Spin}(1,2)\cong\mathrm{SL}(2,\mathbb{R})$. $\mathcal{C}^{\perp}$ is a transverse euclidean CFT which is invariant under the action of Γ. More explicitly, writing the lorentzian metric for $\mathbb{R}^{1,2}$ as $ds^2=-2dx^+dx^-+dx^2$, the generator of the orbifold group Γ acts as

$$X=\begin{pmatrix}x^+\\ x\\ x^-\end{pmatrix}\quad\rightarrow\quad g_0\cdot X=e^{2\pi\mathcal{J}}X=\begin{pmatrix}x^+\\ x+2\pi x^+\\ x^-+2\pi x+\frac{1}{2}(2\pi)^2x^+\end{pmatrix}\;;\qquad \mathcal{J}=\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}\tag{2.1}$$

That is, $g_0=\exp(2\pi iJ)$ where we take the Lie algebra generator

$$J=J^{+x}=\frac{1}{\sqrt{2}}(J^{0x}+J^{1x})\tag{2.2}$$

corresponding to a linear combination of a boost and a rotation.

For some purposes it is convenient to use coordinates [1]

$$\begin{aligned}x^+&=y^+\\ x&=y^+y\\ x^-&=y^-+\frac{1}{2}y^+y^2\end{aligned}\tag{2.3}$$

in terms of which the metric becomes

$$ds^2=-2dy^+dy^-+(y^+)^2dy^2\tag{2.4}$$

and the orbifold identification is simply $y\sim y+2\pi n$. The spacetime (2.4) may be visualized as two cones (parametrized by y^+ and y) with a common tip at $y^+=0$, crossed with the real line (for y^-). This description is not valid at the singularity $x^+=y^+=0$ where this coordinate system is not valid.

The parabolic orbifold has a few attractive features which fit nicely with the criteria outlined in the introduction:

1. The orbifold has a null isometry which allows us to use light-cone evolution. There is no string or particle production in the second quantized theory.

[1] This is a special case of the transformation from "Brinkman" to "Rosen" coordinates in the theory of pp waves [27].

2. The orbifold has a covariant spinor and thus preserves half of the supersymmetries in the superstring theory. As a result, one can show that one-loop cosmological constant and the massless tadpoles vanish.

3. There are *no* closed time-like curves, while there are closed null curves only in the hyperplane $x^+ = 0$.

4. In terms of light-cone time, the orbifold describes the big crunch of a circle at $x^+ = 0$, followed by a big bang. There is a null singular line at $x^+ = 0$. This provides an interesting toy model for understanding the big crunch/big bang singularity in cosmology. It is also related to the singularity of the massless BTZ black hole [28, 29].

5. If the spacetime does not end at the singularity, one might be able to define an S-matrix as a natural observable.

In [15] we observed that there are divergences in the S-matrix of the parabolic orbifold for special kinematic configurations. We suggested there that they might signal the breakdown of perturbation theory. Below we will analyze these divergences in more detail, and will argue that this conclusion is unavoidable. Therefore, even if such an S-matrix exists, it cannot be computed in perturbation theory.

In this paper we consider a few generalizations of the parabolic orbifold by allowing Γ to act nontrivially on the transverse spacetime $\mathcal{C}^\perp$ or by including more generators, in a way that the above attractive features remain. In the following we will discuss in detail a simple family of orbifolds of which the parabolic orbifold is a special example. In particular we will show that this more general family does allow a perturbative computation of the S-matrix. Other generalizations and classifications will be discussed in section 7.

We start with flat four dimensional spacetime $\mathbb{R}^{1,3}$ with the metric

$$ds^2 = -2dx^+dx^- + dx^2 + dz^2 \,, \tag{2.5}$$

and consider the orbifolds obtained by identifying

$$X \sim e^{2\pi n \mathcal{J}} X \,, \qquad z \sim z + 2\pi n R \,, \qquad n \in \mathbb{Z} \tag{2.6}$$

where the column vector X and matrix $\mathcal{J}$ are the same as in (2.1). The generator of the identification is now given by

$$g_0 = e^{2\pi i (J + R p_z)} \,, \tag{2.7}$$

where J is given by (2.2) and p_z is the generator for translation in z direction.

The above orbifold, called the null-brane, was introduced in [4, 11]. The null-brane has a continuous modulus R and the parabolic orbifold corresponds to the singular limit $R \to 0$. The geodesic distance between a point and its n'th image is $2\pi n \sqrt{R^2 + (x^+)^2}$. At finite R the spacetime is regular and Hausdorff everywhere with no closed causal curves. Thus it may also be considered as a regularization of the parabolic orbifold.

The orbifold action breaks the Poincaré symmetry of $\mathbb{R}^{1,3}$, leaving only three of its Lie algebra generators unbroken. These are J, $p^+ = -p_-$ which shifts x^- by a constant, and p_z which generates translations in the z-direction. The null Killing vector associated

JHEP10(2002)031

with p^+ allows us to use a light-cone evolution treating x^+ as time. Thus again there is no particle production in the second quantized theory. As in the parabolic orbifold, for an appropriate choice of the sign of the generator (2.7) acting on spinors, the group Γ leaves one half of all spinor components invariant. When the superstring is compactified on a null-brane it preserves half of the supercharges. These supercharges square to the Killing vector p^+.

Let us now look at the geometry of the null-brane in more detail. We will change to two different coordinate systems. First, for $x^+ \neq 0$ it is convenient to define $y^\pm, y$ as in (2.3) and to express the metric (2.5) as

$$ds^2 = -2dy^+dy^- + (y^+)^2dy^2 + dz^2 \tag{2.8}$$

and the orbifold identification (2.6) becomes

$$\begin{pmatrix} y^+ \\ y \\ y^- \\ z \end{pmatrix} \sim \begin{pmatrix} y^+ \\ y + 2\pi n \\ y^- \\ z + 2\pi nR \end{pmatrix} \tag{2.9}$$

If we also define

$$z = Ry + u \tag{2.10}$$

where u is a noncompact coordinate, then the metric becomes

$$\begin{aligned} ds^2 &= -2dy^+dy^- + du^2 + \left(R^2 + (y^+)^2\right) dy^2 + 2Rdu\, dy & (2.11)\\ &= -2dy^+dy^- + \frac{(y^+)^2}{R^2 + (y^+)^2}du^2 + (R^2 + (y^+)^2)\left(dy + \frac{R}{R^2 + (y^+)^2}du\right)^2 & (2.12)\end{aligned}$$

(2.12) represents a circle fibration over a three-dimensional manifold parameterized by (y^+, y^-, u), where the the circle has a radius $\sqrt{R^2 + (y^+)^2}$ and the fibration has a connection with

$$F = -\frac{Ry^+}{(R^2 + (y^+)^2)^2}dy^+ \wedge du\,. \tag{2.13}$$

It is manifest in the first line of (2.12) that as $R \to 0$ we get the metric of the parabolic orbifold in y coordinate times a transverse noncompact direction

$$ds^2 = -2dy^+dy^- + (y^+)dy^2 + du^2\,. \tag{2.14}$$

Note that since the coordinate transformation (2.3) is singular at $y^+ = 0$, the $y^+ \to 0$ limit of (2.12) should be treated with caution.

Another set of global coordinates [4] can be obtained by taking

$$z = R\theta\,, \qquad \widetilde{X} = e^{-\frac{z}{R}\mathcal{J}}X\,, \tag{2.15}$$

i.e. explicitly,

$$\begin{aligned} x^+ &= \widetilde{x}^+ \\ x &= \widetilde{x} + \theta\widetilde{x}^+ \\ x^- &= \widetilde{x}^- + \theta\widetilde{x} + \frac{1}{2}\theta^2\widetilde{x}^+\,. \end{aligned} \tag{2.16}$$

In these coordinates the metric becomes

$$
\begin{aligned}
ds^2 &= -2d\widetilde{x}^+ d\widetilde{x}^- + d\widetilde{x}^2 + \left(R^2 + (\widetilde{x}^+)^2\right) d\theta^2 + 2d\theta(\widetilde{x}^+ d\widetilde{x} - \widetilde{x} d\widetilde{x}^+) \\
&= -2d\widetilde{x}^+ d\widetilde{x}^- + d\widetilde{x}^2 - \frac{1}{R^2 + (\widetilde{x}^+)^2}(\widetilde{x}^+ d\widetilde{x} - \widetilde{x} d\widetilde{x}^+)^2 + \\
&\quad + \left(R^2 + (\widetilde{x}^+)^2\right)\left(d\theta + \frac{1}{R^2 + (\widetilde{x}^+)^2}(\widetilde{x}^+ d\widetilde{x} - \widetilde{x} d\widetilde{x}^+)\right)^2 .
\end{aligned}
\tag{2.17}
$$

Note that the above metric has a constant determinant $-R^2$. (2.17) represents a circle fibration over a 3-manifold parametrized by $\widetilde{X}$. The latter has nonsingular metric with $\det g = \frac{R^2}{R^2+(\widetilde{x}^+)^2}$. Moreover there is a connection on the circle fibration with

$$
F = \frac{2R^2}{\left(R^2 + (\widetilde{x}^+)^2\right)^2} d\widetilde{x}^+ \wedge d\widetilde{x} . \tag{2.18}
$$

There is a strong formal similarity between this solution and the standard Melvin universe. As in the parabolic orbifold, $\widetilde{x}^+$ plays a dual role of a time and a "radial variable." Note that in the present case it is clear that $-\infty < \widetilde{x}^+ < +\infty$, so that we have the "two-cone" theory. It would be of some interest to extend the present discussion to the analog of the two-parameter Melvin solutions [30]–[32] with nontrivial H-flux but we will not attempt this in the present paper.

The coordinate system (y^μ, z) is not geodesically complete. Its completion is given by the system $(\widetilde{X}, \theta)$ related by

$$
\begin{aligned}
\widetilde{x}^+ &= y^+ \\
\widetilde{x} &= y^+ \left(y - \frac{z}{R}\right) \\
\widetilde{x}^- &= y^- + \frac{1}{2} y^+ \left(y - \frac{z}{R}\right)^2 \\
\theta &= \frac{z}{R} .
\end{aligned}
\tag{2.19}
$$

The $R \to 0$ limit is somewhat subtle in these coordinates since the θ coordinate used in (2.15) and (2.16) is not well defined in this limit.

In terms of light-cone time x^+, the null-brane describes an infinite size circle in the far past shrinking to a minimal radius R at $x^+ = 0$ and then expanding to infinite size in the remote future. An interesting feature of the causal structure of the parabolic orbifold is that every point P with $y^+ > 0$ is always in the causal future of the every point $\widetilde{P}$ with $y^+ < 0$. In the null-brane geometry, while this is no longer so, there is a close analogue. The geodesic distance square between a point $\mathcal{P}_1$ and the n^{th} image of a point $\mathcal{P}_2$ is

$$
\Delta_n(\mathcal{P}_1, \mathcal{P}_2) = -2\Delta y^+ \Delta y^- + y_1^+ y_2^+ (\Delta y - 2\pi n)^2 + (\Delta z - 2\pi n R)^2 \tag{2.20}
$$

The large n behavior of the above equation for fixed $\mathcal{P}_1, \mathcal{P}_2$ is

$$
\Delta_n (\mathcal{P}_1, \mathcal{P}_2) \sim \left(y_1^+ y_2^+ + R^2\right) (2\pi n)^2 \tag{2.21}
$$

Thus, points with $y_1^+ y_2^+ + R^2 < 0$ are necessarily timelike separated.

3. Solutions of the wave equation on the null-brane

In [15] we discussed the solutions of the wave equation on the parabolic orbifold. Here we give an analogous treatment for the null-brane. These functions, which are easily derived from those of the parabolic orbifold, are important for studying first quantized particles on this spacetime and for constructing the vertex operators in string theory.

We start by considering functions on the covering space $\mathbb{R}^{1,3}$. We diagonalize the Killing vectors J, p^+, p_z and the laplacian :

$$\begin{aligned} &\left[-2\frac{\partial}{\partial x^+}\frac{\partial}{\partial x^-}+\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right]\psi = m^2\psi \\ &\widehat{J}\psi = -i\left(x^+\frac{\partial}{\partial x}+x\frac{\partial}{\partial x^-}\right)\psi = J\psi \\ &\widehat{p}^+\psi = i\frac{\partial}{\partial x^-}\psi = p^+\psi\,, \qquad \widehat{p}_z\psi = -i\frac{\partial}{\partial z}\psi = k\psi\,. \end{aligned} \tag{3.1}$$

Following the discussion in [15], the eigenfunctions functions can be written as

$$\begin{aligned} \psi_{p^+,J,k,m^2} &= \sqrt{\frac{1}{ix^+}}\exp\left[-ip^+x^- - i\frac{m^2+k^2}{2p^+}x^+ + i\frac{p^+}{2x^+}(x-\xi)^2 + ikz\right] \\ &= \int_{-\infty}^{\infty}\frac{dp}{\sqrt{2\pi p^+}}e^{-ip\xi}\exp\left[-ip^+x^- - i\frac{p^2+k^2+m^2}{2p^+}x^+ + ipx + ikz\right] \end{aligned} \tag{3.2}$$

with

$$\xi = -\frac{J}{p^+}\,. \tag{3.3}$$

The second line of (3.2) shows that J-eigenfunctions can be obtained from the standard momentum eigenfunctions by a Fourier transform. The functions ψ_{p^+,J,k,m^2} form a complete basis of functions for fixed x^+ with the inner product

$$\int_{-\infty}^{\infty} dx^- dx dz \psi^*_{p^+,J,k,m^2}\psi_{\widetilde{p}^+,\widetilde{J},\widetilde{k},m^2} = (2\pi)^3\delta(p^+-\widetilde{p}^+)\delta(J-\widetilde{J})\delta(k-\widetilde{k})\,. \tag{3.4}$$

Consider now the null-brane orbifold $\mathcal{O} = \mathbb{R}^{1,3}/\Gamma$. Under the identification (2.6) the function (3.2) transforms as

$$\psi_{p^+,J,k,m^2} \to e^{2\pi in(J+kR)}\psi_{p^+,J,k,m^2}\,. \tag{3.5}$$

It then follows that in the null-brane functions should satisfy the quantization condition

$$J + kR = n \in \mathbb{Z}\,. \tag{3.6}$$

Therefore, we label the basis functions

$$V_{p^+,J,n,m^2} = \psi_{p^+,J,k=\frac{n-J}{R},m^2}\,. \tag{3.7}$$

The orbifold $\mathcal{O}$ is foliated by equal x^+ spaces $\mathcal{F}_{x^+}$ and the complete basis functions V_{p^+,J,n,m^2} satisfy

$$\int_{\mathcal{F}_{x^+}} V^*_{p^+,J,n,m^2}V_{\widetilde{p}^+,\widetilde{J},\widetilde{n},m^2} = (2\pi)^3 R\delta_{n,\widetilde{n}}\delta\left(p^+-\widetilde{p}^+\right)\delta\left(J-\widetilde{J}\right)\,. \tag{3.8}$$

JHEP10(2002)031

In terms of the coordinate system (2.3)(2.10) the functions (3.7) can be written as

$$V_{p^+,J,n,m^2} = \sqrt{\frac{1}{iy^+}}\exp\left[-ip^+y^- + iny + i\frac{J^2}{2p^+y^+} - i\frac{\left(\frac{n-J}{R}\right)^2 + m^2}{2p^+}y^+ + i\left(\frac{n-J}{R}\right)u\right] \tag{3.9}$$

and in terms of the coordinates (2.15) they are

$$V_{p^+,J,n,m^2} = \sqrt{\frac{1}{i\widetilde{x}^+}}\exp\left[-ip^+\widetilde{x}^- - i\frac{\left(\frac{n-J}{R}\right)^2 + m^2}{2p^+}\widetilde{x}^+ + i\frac{p^+}{2\widetilde{x}^+}(\widetilde{x}-\xi)^2 + in\theta\right] \tag{3.10}$$

(to derive this, it is convenient to use $\psi_J(X) = e^{\theta J}\,\psi_J(\widetilde{X})$).

The basis functions (3.7) are singular at $x^+ = 0$. More explicitly, we have

$$\lim_{x^+\to 0} V_{p^+,J,n,m^2}(x^+,x,x^-,z) = \sqrt{\frac{2\pi}{p^+}}e^{-ip^+x^- + i\left(\frac{n-J}{R}\right)z}\delta(x-\xi) = \sqrt{\frac{2\pi}{p^+}}e^{-ip^+\widetilde{x}^- + in\theta}\delta(\widetilde{x}-\xi)\,. \tag{3.11}$$

This implies that these basis functions are localized at $\widetilde{x} = x = \xi$ at $x^+ = 0$ [15].

These functions are potentially dangerous to use as vertex operators for two related reasons:

1. Focusing the particles at a point results in infinite energy density and can cause large back reaction when the system is coupled to gravity (as in string theory).

2. The expression for the basis functions in terms of plane waves as in the second expression in (3.2) involves integrating over arbitrarily high energies. This leads to certain divergences in S-matrix elements.

The origin of these problems can be traced back to the diagonalization of $\widehat{J}$, and one might expect that they can therefore be avoided by smearing them with smooth functions of J of rapid decrease.[2] We now demonstrate that this is the case. Consider the wave packets

$$U_{p^+,f(J),n,m^2} = \int dJ f(J) V_{p^+,J,n,m^2} \tag{3.12}$$

where $f(J)$ is a smooth function of rapid decrease. Their inner product is

$$\int_{\mathcal{F}_{x^+}} U^*_{p^+,f,n,m^2} U_{\widetilde{p}^+,\widetilde{f},\widetilde{n},m^2} = (2\pi)^3 R\delta\left(p^+ - \widetilde{p}^+\right)\delta_{n,\widetilde{n}}\int dJ f(J)^*\widetilde{f}(J)\,. \tag{3.13}$$

Since V is bounded for $x^+ \neq 0$, the integral over J in (3.12) converges for all $x^+ \neq 0$. Moreover the singularity at $x^+ = 0$ is smoothed out

$$\begin{aligned}\lim_{x^+\to 0} U_{p^+,f,n,m^2}(x^+,x,x^-,z) = &= \sqrt{2\pi p^+}\exp\left(-ip^+x^- + i(n+xp^+)\frac{z}{R}\right)f(-xp^+) = \\ &= \sqrt{2\pi p^+}e^{-ip^+\widetilde{x}^- + in\theta}f(-\widetilde{x}p^+)\,.\end{aligned} \tag{3.14}$$

[2] Recall that the Schwarz space of functions of rapid decrease is the space of functions with $\lim_{J\to\pm\infty}|J^n(\frac{d}{dJ})^m f(J)| = 0$ for any n, m. This space is preserved by Fourier transform [33].

Therefore, U does not have any singularities. This solves the first problem mentioned above.

Using (3.2)(3.7)(3.12) we can express U as an integral over plane waves

$$U_{p^+,f,n,m^2} = \int \frac{dJdp}{\sqrt{2\pi p^+}} f(J) \exp\left[ip\frac{J}{p^+} - ip^+x^- - i\frac{p^2 + (\frac{n-J}{R})^2 + m^2}{2p^+}x^+ + ipx + i\left(\frac{n-J}{R}\right)z \right]. \tag{3.15}$$

Since $f(J)$ is of rapid decrease its Fourier transform is of rapid decrease. Therefore, in (3.15) the support of wave functions with large energy $\frac{1}{2p^+}\left(p^2 + \left(\frac{n-J}{R}\right)^2 + m^2\right)$ is suppressed.[3]

It is important that in the parabolic orbifold the label J is discrete. Therefore we cannot form wave packets as here and smear the singularity. Note that by integrating over p^+ the singularities of the wave functions with $J \neq 0$ can be smeared. However, there is no way to avoid the singularity of the functions with $J = 0$.

To demonstrate this general discussion of wavefunctions in the null-brane, consider the wave packet with $f(J) = e^{-\frac{J^2}{2}}$,

$$U_{p^+,-\frac{J^2}{2},n,m^2} = \int dJ e^{-\frac{J^2}{2}} V_{p^+,J,n,m^2}\,. \tag{3.16}$$

It is simple to express the above function in $\widetilde{x}$ coordinates

$$U_{p^+,-\frac{J^2}{2},n,m^2} = \sqrt{\frac{2\pi p^+}{K}} \exp\left[-\frac{Q}{2K}\right] \exp\left(-ip^+\widetilde{x}^- - i\frac{m^2}{2p^+}\widetilde{x}^+ + in\theta\right) \tag{3.17}$$

where

$$\begin{aligned} K &= p^+\widetilde{x}^+ + i\left[\left(\frac{\widetilde{x}^+}{R}\right)^2 - 1\right], \\ Q &= \left(\frac{p^+\widetilde{x}^+}{R^2} - i(p^+)^2\right)\widetilde{x}^2 + \left[\frac{\widetilde{x}^+}{p^+R^2} - i\left(\frac{\widetilde{x}^+}{R}\right)^2\right]n^2 + \frac{2\widetilde{x}^+\widetilde{x}n}{R^2}\,. \end{aligned} \tag{3.18}$$

It is manifest in (3.17) and (3.18) that U is completely regular everywhere. It is also instructive to write U as a wave packet in momentum space using the second line of (3.2). We find that

$$U_{p^+,-\frac{J^2}{2},n,m^2} = \sqrt{\frac{1}{p^+ + i\frac{\widetilde{x}^+}{R^2}}} \exp\left(-ip^+\widetilde{x}^- - i\frac{m^2 + \frac{n^2}{R^2}}{2p^+}\widetilde{x}^+ + in\theta\right) \int d\widetilde{p}\, g(\widetilde{p}) e^{i\widetilde{p}\widetilde{x} - i\frac{\widetilde{p}^2}{2p^+}\widetilde{x}^+} \tag{3.19}$$

with

$$g(\widetilde{p}) = \exp\left[-\frac{1}{2\left(1 + i\frac{\widetilde{x}^+}{p^+R^2}\right)} \left(\frac{\widetilde{p}}{p^+} + \frac{\widetilde{x}^+ n}{p^+R^2}\right)^2 \right]\,. \tag{3.20}$$

Thus U corresponds to a gaussian wave packet in $\widetilde{p}$ which is the momentum conjugate to $\widetilde{x}$. In particular, the higher energy region of the integration is suppressed.

[3] More precisely, this statement is true only for fixed $(x^{\pm}, x, z)$, and there is no universal bound for all values of $(x^{\pm}, x, z)$.

JHEP10(2002)031

The $R \to 0$ limit of the null-brane is the parabolic orbifold times a noncompact line parametrized by z. In order for the function V_{p^+,J,n,m^2} to have a sensible limit we should hold the momentum along the z direction, $k = \frac{n-J}{R}$, fixed as $R \to 0$. The resulting function is ψ_{p^+,J,k,m^2} of (3.2) with $J = n \in \mathbb{Z}$, which is a good function on the parabolic orbifold times the line. In order to have a good limit for the wave packets U (3.12) we should let the function $f(J)$ depend also on n and R, $f_n(J,R)$, such that $\lim_{R\to 0} f_n(J = n - kR, R) = \widetilde{f}_n(k)/R$ with finite $\widetilde{f}_n(k)$. Then

$$\lim_{R\to 0} U = \int dk \widetilde{f}(k) \psi_{p^+,J=n,k,m^2} \tag{3.21}$$

i.e. it has fixed value for $J = n \in \mathbb{Z}$ and is a wave packet in z. Clearly, this function diverges at $x^+ = 0$.

One can work out the propagator for a massive particle on the null-brane. It can be obtained from the propagators in $\mathbb{R}^{1,3}$ by summing over the images under the orbifold action, leading to

$$G_F(\mathcal{P}_1, \mathcal{P}_2) = \frac{i}{8\pi} \sum_{n\in\mathbb{Z}} \left(\frac{m^2}{\Delta_n + i\epsilon} \right)^{1/2} H_1^{(2)} \left(\sqrt{m^2(\Delta_n + i\epsilon)} \right) \tag{3.22}$$

where Δ_n is the invariant distance square between a point $\mathcal{P}_1$ and the n^{th} image of $\mathcal{P}_2$ (2.20). Note that the propagator for the parabolic orbifold is also given by (3.22) with $R = 0$ in (2.20).

The propagator (3.22) can be expanded in terms of the wave functions discussed in this section,

$$G(\mathcal{P}_1, \mathcal{P}_2) = \theta(\Delta y^+) D(\mathcal{P}_1, \mathcal{P}_2) + \theta(-\Delta y^+) D(\mathcal{P}_2, \mathcal{P}_1) \tag{3.23}$$

where when $\Delta y^+ > 0$

$$\begin{aligned} D(\mathcal{P}_1, \mathcal{P}_2) = &\int_0^\infty \frac{dp^+}{(2\pi)^3 2p^+ R} \times \\ &\times \int_{-\infty}^{\infty} dJ \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{iy_1^+}\sqrt{-iy_2^+}} \exp\Bigg[-ip^+ \Delta y^- + iJ\Delta y + i\frac{n-J}{R}\Delta z - \\ &\qquad - i\frac{J^2}{2p^+}\left(-\frac{1}{y_1^+} + \frac{1}{y_2^+} \right) - i\frac{m^2 + \left(\frac{n-J}{R}\right)^2}{2p^+}\Delta y^+ \Bigg]. \end{aligned} \tag{3.24}$$

Note that the sum over n in (3.24) and (3.22) are not the same. They are related by a Poisson resumation.

4. String quantization on the null-brane

In this section we consider the canonical quantization of strings in the null-brane background. The discussion is very similar to that of the parabolic orbifold in [15] and therefore it will be very brief.

In the covariant formalism we use four free fields $x^{\pm}, x, z$. In the w twisted sector they are subject to the twisted boundary conditions

$$\begin{pmatrix} x^+(\sigma+2\pi,\tau) \\ x(\sigma+2\pi,\tau) \\ x^-(\sigma+2\pi,\tau) \\ z(\sigma+2\pi,\tau) \end{pmatrix} = g_0^w \cdot \begin{pmatrix} X \\ z \end{pmatrix} = \begin{pmatrix} x^+(\sigma,\tau) \\ x(\sigma,\tau) + 2\pi w x^+(\sigma,\tau) \\ x^-(\sigma,\tau) + 2\pi w x(\sigma,\tau) + \frac{1}{2}(2\pi w)^2 x^+(\sigma,\tau) \\ z(\sigma,\tau) + 2\pi w R \end{pmatrix}. \tag{4.1}$$

These can be "solved" using free fields [15] and lead to an interesting exchange algebra similar to exchange algebras in RCFT. [4]

In the lightcone gauge we have two possible procedures:

1. Use the original $x^{\pm}, x, z$ coordinates. The advantage of this method is that the worldsheet lagrangian is free. The somewhat unusual aspect of this procedure is that the worldsheet hamiltonian p_x^- is not invariant under the orbifold action which is a gauge symmetry of the system. Also, in the twisted sectors the periodicity rules of the fields depend on the worldsheet time $x^+ = \tau$.

2. Use an invariant hamiltonian. This can be done either with the coordinates (2.3), (2.10), where the hamiltonian is $p_y^- = i\partial_{y^+}$ or with the coordinates (2.15), where the hamiltonian is $p_{\widetilde{x}}^- = i\partial_{\widetilde{x}^+}$. The disadvantage of this procedure is that unlike the first one, the system is time (x^+) dependent.

It is essential to keep as dynamical variables in the lagrangian the zero modes of x^- in the first procedure, and the zero mode of y^- or $\widetilde{x}^-$ in the second procedure [15, 34]. In the first procedure this coordinate is needed to ensure the invariance of the system under the orbifold action. In the second procedure this is needed in order to absorb all infinite renormalization constants in appropriate counter terms.

For brevity here we will follow only the first procedure. The light-cone gauge lagrangian is

$$L = -p^+\partial_\tau x_0^- + \frac{1}{4\pi\alpha'}\int_0^{2\pi} d\sigma \left(\alpha' p^+ \partial_\tau x \partial_\tau x - \frac{1}{\alpha' p^+}\partial_\sigma x \partial_\sigma x + \alpha' p^+ \partial_\tau z \partial_\tau z - \frac{1}{\alpha' p^+}\partial_\sigma z \partial_\sigma z\right). \tag{4.2}$$

Invariance under constant shifts of σ is implemented by imposing [34]

$$\int d\sigma \left(\partial_\sigma x \partial_\tau x + \partial_\sigma z \partial_\tau z - \frac{1}{2\tau}\partial_\sigma x^2\right) = 0\,. \tag{4.3}$$

It is important that the lagrangian (4.2) and the expressions for the constraint (4.3) are invariant under the orbifold identification

$$\begin{aligned} x(\sigma,\tau) &\to x(\sigma,\tau) + 2\pi n\tau \\ x_0^-(\tau) &\to x_0^-(\tau) + 2\pi n \int_0^{2\pi}\frac{d\sigma}{2\pi} \\ x(\sigma,\tau) + \frac{(2\pi n)^2}{2}\tau z(\sigma,\tau) &\to z(\sigma,\tau) + 2\pi n R\,. \end{aligned} \tag{4.4}$$

[4] Although there is a potential role for noncommutative geometry here, note that the spacetime is perfectly Hausdorff, as is the WZW theory.

The equation of motion for x_0^- sets p^+ to a constant. The equation of motion for p^+ leads to

$$P_{x^-} = p^+ \partial_\tau x_0^- = \frac{1}{4\pi\alpha'} \int_0^\ell d\sigma \left(\partial_\tau x \partial_\tau x + \partial_\sigma x \partial_\sigma x + \partial_\tau z \partial_\tau z + \partial_\sigma z \partial_\sigma z\right) \tag{4.5}$$

where we have rescaled σ to range in $[0, \ell = 2\pi\alpha' p^+)$.

A complete set of solutions to the equations of motion in the w-twisted sector, can be expressed in terms of harmonic oscillators:

$$\begin{aligned} x(\sigma,\tau) &= -\frac{J}{p^+} + \frac{p}{p^+}\tau + \frac{2\pi w\sigma\tau}{\ell} i \left(\frac{\alpha'}{2}\right)^{\frac{1}{2}} \times \\ &\quad \times \sum_{n\neq 0} \left\{ \frac{\alpha_n}{n} \exp\left[-\frac{2\pi i n(\sigma+\tau)}{\ell}\right] + \frac{\widetilde{\alpha}_n}{n} \exp\left[\frac{2\pi i n(\sigma-\tau)}{\ell}\right]\right\} \\ z(\sigma,\tau) &= z_0 + \frac{k}{p^+}\tau + \frac{2\pi w R\sigma}{\ell} \left(\frac{\alpha'}{2}\right)^{\frac{1}{2}} \times \\ &\quad \times \sum_{n\neq 0} \left\{ \frac{\alpha^z_n}{n} \exp\left[-\frac{2\pi i n(\sigma+\tau)}{\ell}\right] + \frac{\widetilde{\alpha}^z_n}{n} \exp\left[\frac{2\pi i n(\sigma-\tau)}{\ell}\right]\right\}. \end{aligned} \tag{4.6}$$

The solution of x_0^- is obtained from (4.5). Upon quantization these oscillators obey the standard canonical commutation relations and $n = J + kR$, $w \in \mathbb{Z}$.

As in [15], it is straightforward to extend the worldsheet lagrangian (4.5) to the Green-Schwarz formalism. For concreteness consider the model on $\mathcal{O} \times \mathbb{R}^6$. Before taking the quotient by Γ we should add to (4.5) six free worldsheet bosons x^i and eight rightmoving fermions S^a (in the type II theory we also need eight leftmoving fermions and in the heterotic string also leftmoving degrees of freedom for the internal degrees of freedom). It is easy to see using the symmetries of the problem that after the action by Γ the added fields x^i and S^a remain free. The boundary conditions of these fields depend on the spin structure around the nontrivial cycle. For the spin structure which preserves supersymmetry x^i and S^a are periodic around the string. For the other spin structure S^a transforms with $(-1)^w$, as in the supersymmetry breaking compactification of [35].

5. Torus partition function

In the one-loop amplitudes we sum over the "sectors"

$$\begin{aligned} (X,z)(\sigma^1+1,\sigma^2) &= \left(e^{2\pi w^a \mathcal{J}} X, z + 2\pi R w^a\right)(\sigma^1,\sigma^2) \\ (X,z)(\sigma^1,\sigma^2+1) &= \left(e^{2\pi w^b \mathcal{J}} X, z + 2\pi R w^b\right)(\sigma^1,\sigma^2)\,. \end{aligned} \tag{5.1}$$

Comparing with the analogous calculation in [15], the only difference is the replacement $(x^+)^2 \to (x^+)^2 + R^2$ as the effective radius-square in equation (5.13) of [15]. That is, the contribution of the classical action to the torus amplitude is given by

$$\exp\left[-\frac{\pi\left[(x^+)^2 + R^2\right]}{\alpha'} \frac{|w_b + w_a\tau|^2}{\tau_2}\right]. \tag{5.2}$$

Equation (5.2) is most easily derived using the coordinate system $(\widetilde{X}, \theta)$ since the classical configuration is then given by

$$\theta\left(\sigma^1, \sigma^2\right) = 2\pi \left(w^a \sigma^1 + w^b \sigma^2\right) \tag{5.3}$$

with $\widetilde{X}$ a constant.

We would like to make a few comments about the torus amplitude:

1. Both in the parabolic orbifold and in the null-brane we can study the system either with supersymmetric boundary conditions around the nontrivial cycle or with nonsupersymmetric boundary conditions as in [35]. With supersymmetric boundary conditions we find cancellations and the one loop cosmological constant vanishes.

2. The final answer is a cosmological constant as a function of $(x^+)^2 + R^2$. Regardless of the boundary conditions as $R \to \infty$ we recover the standard flat space answer because in this limit our spacetime becomes time independent. Similarly, in the large lightcone time limit $x^+ \to \pm\infty$ the cosmological constant vanishes in the superstring theory.

3. As $R \to 0$ we recover the results of [15] for the parabolic orbifold.

4. If R is sufficiently small, $(x^+)^2 + R^2$ can be smaller than order α'. Then if supersymmetry is broken the system can have tachyons in the winding sector and the cosmological constant can diverge.

5. As $x^+ \to 0$ for finite R we do not find a continuum of winding modes because the T-dual space is of finite size (α'/R).

6. Tree amplitudes

In this section we consider the tree-level amplitudes for the untwisted modes in the null-brane. We will show that the tree-level amplitudes for the wave packets U of (3.12) in the null-brane are better behaved than the amplitudes studied in [15]. The amplitudes involving twisted modes, which are important for understanding twisted mode production, will be left for future work.

As in [15], we calculate the tree-level untwisted amplitudes in the null-brane using the inheritance principle from those in flat space in the J-basis by restricting $n = J + kR$ to integers. Therefore, in the J-basis the tree level untwisted S-matrix is essentially the same in flat space, in the parabolic orbifold and in the null-brane. The amplitudes in the J-basis in flat spacetime are in turn computed from those in the momentum basis by a Fourier transform, thanks to the simple relation between the J-basis functions and the plane waves. For simplicity we will only look at tachyon amplitudes.

The vertex operator for the tachyon is given by

$$V_{p^+,J,n,p_\perp}(\sigma,\tau) = \frac{1}{\sqrt{2\pi p^+}} \int_{-\infty}^{\infty} dp\, e^{ip\xi}\, e^{i\vec{p}\cdot\vec{X}(\sigma,\tau)}\,, \qquad \xi = -\frac{J}{p^+} \tag{6.1}$$

where

$$e^{i\vec{p}\cdot\vec{X}(\sigma,\tau)} = \exp\left[-ip^+x^- - ip^-x^+ + ipx + ikz + i\vec{p}_\perp\cdot\vec{x}_\perp(\sigma,\tau)\right] \tag{6.2}$$

($\vec{p}_\perp$ and $\vec{x}_\perp$ denote vectors in other transverse dimensions) is the standard on-shell tachyon vertex operator with

$$p^- = \frac{p^2+\mathrm{m}^2}{2p^+}\,, \qquad \mathrm{m}^2 = m^2 + k^2 + \vec{p}_\perp^{\,2}\,, \qquad m^2 = -\frac{4}{\alpha'}\,, \qquad k = \frac{n-J}{R}\,. \tag{6.3}$$

It then follows that in the null-brane the three- and four-point amplitudes in terms of the above vertex operators are given precisely by equations (6.4)–(6.6) and (6.13)–(6.17) of [15] with the substitution

$$(\vec{p}_{\perp i})_{\text{there}} = (k_i, \vec{p}_{\perp i})\,, \qquad k_i = \frac{n_i - J_i}{R}\,. \tag{6.4}$$

Thus the behavior of the four-point amplitudes are the same as those discussed in [15], which involve integrating the momentum space amplitudes over an infinite range of the Mandelstam variable s. An important question is whether the S-matrix is finite, since the integrals involve very large s, i.e. potentially infinite center of mass energy. Roughly, the contribution of the large s region of the integral to the amplitudes can be written as

$$A_J \approx I_+ + I_-$$

with

$$I_\pm \sim \int_{-\infty}^{\infty} \frac{d\sigma}{|\sigma|} e^{i\sigma F_\pm(J_i,p_i^+)} A_{VS}\left(s(\sigma), t_\pm(\sigma), u_\pm(\sigma)\right) \tag{6.5}$$

where A_{VS} is the Virasoro-Shapiro amplitude in momentum space, $F_\pm(J_i, p_i^+)$ are functions of J and p^+ of the external particles, and

$$s = (p_1^+ + p_2^+)\sigma^2 + \left(\sqrt{\frac{p_1^+}{p_2^+}}k_1 - \sqrt{\frac{p_2^+}{p_1^+}}k_2\right)^2 + \mu_s(\vec{p}_{\perp i}, p_i^+)\,. \tag{6.6}$$

μ_s is a function of $\vec{p}_\perp$ and p_i^+ only, whose detailed form will not concern us below. Note that in (6.5) the Mandelstam variables t, u are different functions of σ in $I_\pm$, given by $t_\pm(\sigma)$ and $u_\pm(\sigma)$ respectively. For more detailed expressions of various functions in (6.5) see [15]. To compare with the expressions in [15], note that σ in (6.5) corresponds to q_+ in equation (6.17) there and $I_\pm$ here correspond to equation (6.16) of [15] in the $|q| \to \infty$ and $|q| \to 0$ limits respectively.

The convergence behavior of (6.5) can be summarized as follows:

1. For generic p_i^+, $A_{VS}(s(\sigma), t_\pm(\sigma), u_\pm(\sigma))$ behaves for large s as in the hard scattering limit of large s, t, u with fixed ratios. In this limit

$$A_{VS} \sim e^{-\lambda s} \tag{6.7}$$

for some positive constant λ. The integral over σ (6.5) converges for all values of J_i and in particular the dependence on J_i is analytic.

JHEP10(2002)031

2. When $p_t^+ = p_3^+ - p_1^+ = 0$, while I_- behaves as in 1. above and is finite, I_+ is divergent for some values of J_i and $\vec{p}_{\perp i}$. More explicitly, as $|\sigma|$ goes to ∞,

$$t_+ \approx -(k_1 - k_3)^2 - (\vec{p}_{\perp 1} - \vec{p}_{\perp 3})^2 = -(k_1 - k_3)^2 - \vec{p}_{\perp t}^2 \tag{6.8}$$

remains finite. Thus the integrand of I_+ in (6.5) is in the Regge scattering limit of large s with finite t. In this limit, the leading terms in I_+ can be written as

$$I_+ \sim \int \frac{d\sigma}{|\sigma|} e^{i\sigma F_+} s^{-\frac{1}{2}\alpha' \mathrm{m}_t^2} \left(\Gamma\left(\frac{\alpha'}{4}\mathrm{m}_t^2\right)\right)^2 \sin\left(\frac{\alpha'\pi}{4}\mathrm{m}_t^2\right) \tag{6.9}$$

where s is given by (6.6) and

$$\alpha' \mathrm{m}_t^2 = -4 - \alpha' t_+ \,, \qquad F_+ = \frac{1}{\sqrt{\mu_{12}}}(J_3 - J_1)\,, \qquad \mu_{12} = \frac{p_1^+ p_2^+}{p_1^+ + p_2^+}\,. \tag{6.10}$$

It then follows that the integral for A_J diverges as $|\sigma| \to \infty$ when

$$p_t^+ = 0\,; \qquad J_t = J_3 - J_1 = 0\,; \qquad \alpha' \vec{p}_{\perp t}^2 + \alpha'(k_1 - k_3)^2 < 4 \tag{6.11}$$

i.e. close to forward scattering (but not only in the forward direction). There is a similar divergence with $3 \leftrightarrow 4$ associated with the u-channel.

In the null-brane, as we discussed in section 3, instead of using singular functions (6.1) as vertex operators, it is more appropriate to form wave packets (3.12) in terms of rapidly decreasing functions of J. We showed that the wave packets are finite everywhere and in particular, the high energy region of the integration is suppressed by a rapidly decreasing function. It is clear that when $p_t^+ = p_3^+ - p_1^+ \neq 0$, by integrating the amplitudes (6.5) over J_i with a kernel $\prod_i f_i(J_i)$ we find a finite answer. The same is true for I_- when $p_t^+ = 0$. For I_+, when $p_t^+ = 0$, the situation is more subtle. First note that since

$$k_i = \frac{n_i - J_i}{R}\,, \qquad n_i \in \mathbb{Z}$$

when we integrate over J_i from $-\infty$ to ∞, effectively we are integrating k_i over the same ranges. At first sight this seems to invalidate our approximation used in (6.9) (t_+ is no longer fixed). However, since $f_i(J_i)$ suppresses large J_i (i.e. large k_i) contribution, we expect that for the purpose of estimating the convergence of the amplitudes, it is still legitimate to use (6.9). Therefore we have the amplitudes

$$A_U \sim \prod_{i=1}^{4} \left(\int dJ_i f_i(J_i)\right) I_+ + \cdots \tag{6.12}$$

where I_+ is given by (6.9) and $\cdots$ denotes other finite contributions including those from I_-. The integrals (6.12) are potentially divergent around $J_t = J_3 - J_1 \approx 0$ when $n_1 = n_3$ and $\vec{p}_{\perp t} \approx 0$. In this region, using (6.9) and noting that $\alpha' t_+ \cong 0$, (6.12) can be approximated as

$$A_U \sim \int dJ_t \int \frac{d\sigma}{|\sigma|} \frac{1}{\frac{J_t^2}{R^2} + \vec{p}_{\perp t}^2} e^{i\frac{\sigma}{\sqrt{\mu_{12}}}J_t} |\sigma|^{4 - \frac{\alpha'}{R^2}J_t^2 - \alpha' \vec{p}_{\perp t}^2} \tag{6.13}$$

for a reasonably general class of rapidly decreasing functions f_i.[5] The factor $\frac{1}{J_t^2/R^2+\vec{p}_{\perp t}^2}$ in (6.13), which comes from the product of the Γ and the sine functions in (6.9), can be understood as arising from a t-channel propagator of a soft graviton. Interestingly, when $\vec{p}_{\perp t}=0$, the integrand of the J_t integral is singular at $J_t=0$ due to both an IR divergence coming from the $1/J_t^2$ factor and a UV divergence coming from the unbounded σ integral.

If we define the integral (6.13) by first integrating over J_t and then over σ we find that it is finite whenever $\vec{p}_{\perp t}\neq 0$. This can be understood from the fact that the potential UV divergence from the σ integral is integrable when we integrate over J_t. However, the familiar $1/\vec{p}_{\perp t}^2$ singularity at $\vec{p}_{\perp t}=0$ is enhanced by the UV region of large σ to a stronger singularity – a larger inverse power of $\vec{p}_{\perp t}^2$ up to logarithmic corrections.

To illustrate this more clearly, let us look at the $\alpha'\to 0$ limit of (6.13). In this case the integrals can be easily evaluated, and we find that the amplitude scale as $1/|\vec{p}_{\perp t}|^5$. α' corrections make the amplitudes less divergent but not finite. A more precise analysis of the singularities would be desirable.

To summarize, we find that in the null-brane, the J-basis amplitudes (6.5) suffer from the same divergences as those in the parabolic orbifold. When we use smooth wave packets (3.12) in terms of rapidly decreasing functions of J, the amplitudes are better behaved, but still have divergences when $p_t^+=n_3-n_1=\vec{p}_{\perp t}=0$.

Discussion of various divergences There are many known examples of divergent S-matrix elements at non-generic kinematics. Some standard examples include singularities associated with on shell intermediate particles. In terms of the integral over the first quantized parameters (moduli) they originate from the regions $x^+\to\pm\infty$, in which intermediate propagators are long. In our case, since our external states are not p^- invariant (corresponding to the lack of p^- invariance of the parabolic orbifold and the null-brane backgrounds), the corresponding singularities in the S-matrix are not poles. Instead, our S-matrix exhibits some other nonanalytic dependence on the external momenta which originates from the large x^+ behavior (IR singularities) [15].

The singularities of the four-point amplitudes in the J-basis when $p_t^+=J_t=0$ and $\vec{p}_{\perp t}^2$ is sufficiently small (as in (6.11)) are IR singularities because $p_t^+=0$, but they also have another crucial element. They arise from a divergence of (6.9) at large s, and therefore are also UV singularities. They originate from the singularity of $V_{p^+,J,n,\vec{p}_\perp}$ at $x^+=0$, or equivalently from the fact that $V_{p^+,J,n,\vec{p}_\perp}$ has arbitrarily large energy. As discussed after equation (3.11), the focusing at $x^+=0$ leads to infinite energy density and therefore large coupling to the graviton, and hence the divergence.

Unlike the standard IR singularities, our singularities occur for a *range* of values of $\vec{p}_{\perp t}^2$ and not only for $\vec{p}_{\perp t}=0$. Therefore, it is impossible to remove them by the standard procedure of dealing with IR singularities, and the results suffer from incurable IR divergences. Presumably this signals breakdown of the perturbation theory.[6] This

[5] Note that (6.13) is valid even when the support of $f_1(J_1)$ is very different from that of $f_3(J_3)$, as long as $f_1 f_3$ does not vanish for $J_1=J_3$.

[6] It would be rather desirable to have a clean computation of 1-loop scattering amplitudes indicating a clear divergence.

breakdown of perturbation theory should perhaps be interpreted physically as follows: If we try to scatter particles which become infinitely focused, the nonlinear backreaction of the metric results in a singularity. The condition $p_t^+ = J_t = 0$ states that the incoming particle 1 and the outgoing particle 3 are focused at the same point at $x^+ = 0$. It is clear that otherwise, the amplitude cannot diverge [15]. For a related recent discussion see [21, 22].

In order to solve this problem we need to prevent the focusing of the incoming particles. This can be done only by scattering wave packets which smear the values of J_1 and J_2.

In flat space this is easily done. For example we can scatter p eigenstates. More generally, a wave packet in the J-basis is equivalent to a wave packet in momentum space by a Fourier transform and therefore the amplitudes are finite.

In the parabolic orbifold since J is discrete we do not have the freedom to integrate over J to form wave packets. This reflects the fact that in the parabolic orbifold, the energy of incoming states is always blue-shifted without bound as they approach the singularity. Therefore, the S-matrix elements in the parabolic orbifold cannot be computed in perturbation theory. It might also mean that the notion of an S-matrix for scattering from $x^+ = -\infty$ to $x^+ = +\infty$ is simply not valid in the interacting theory due to large back reaction effects. It is possible that one should formulate the theory with only one of the two asymptotic regions and interpret vertex operator correlators differently.

Another perspective on the divergence comes from noting that the divergence arises from Regge behavior of the scattering amplitudes. Already in the standard flat space amplitudes Regge behavior is related to a breakdown of string perturbation theory for forward scattering amplitudes. There are plausible resummation techniques for determining the nonperturbative forward amplitudes [36, 37]. If we (naively!) apply the inheritance principle to such resummed amplitudes we find that the divergence is softened.

In the null-brane, the spacetime is smooth everywhere and we expect that, as in flat space, the amplitudes for wave packets should be well-behaved. Indeed, we find, by integrating the amplitudes (6.9) over J_i with a kernel $\prod_i f_i(J_i)$, a finite answer even when $p_t^+ = 0$ for generic $\vec{p}_{\perp i}$. However, when $p_t^+ = n_3 - n_1 = 0$ and $\vec{p}_{\perp t} \to 0$ the amplitudes diverge like an inverse power of $\vec{p}^{\,2}_{\perp t}$. As we have emphasized, this singularity is not merely the standard IR singularity due to the propagator of the soft graviton in the *t-channel*. It is enhanced due to the integration over the UV region of the *s-channel*. It would be desirable to have a clearer understanding of this curious mixing of UV/IR divergences. Unlike the situation in the parabolic orbifold or in the scattering of J eigenstates in the null-brane, which we discussed above, here the singularity occurs only in the forward direction, i.e. for $p_t^+ = \vec{p}_{\perp t} = 0$. We will not try to prove it here, but we suspect that, when there are a certain number of uncompact transverse dimensions, such a singularity may be removed by the standard procedure for dealing with IR divergences due to soft gravitons, and therefore the physical results are sensible.

There is an important special case of the previous discussion in which there are no transverse noncompact dimensions to the null brane. Then $\vec{p}_\perp$ takes discrete values, and therefore the divergence at $\vec{p}_{\perp t}$ cannot be removed. We expect that in this case the singularity is just as harmful as in the parabolic orbifold, and that perturbation theory breaks

JHEP10(2002)031

down. It might come as a surprise that in the null-brane, when there are no other noncompact dimensions, even with a smooth geometry and smooth wave packets, we find divergences in the amplitudes which appear to be incurable. This situation is reminiscent of the singular response of de Sitter space to small perturbations in the infinite past (see e.g. [38, 39]).

Let us return to the null-brane with sufficient number of noncompact dimensions. It is of some interest to understand how the finite S-matrix at $R > 0$ becomes singular for $R \to 0$. As $R \to 0$ the wave packets U have fixed $J = n \in \mathbb{Z}$ and become wave packets in z (3.21). The limit of the S-matrix has the same divergence we encountered in the parabolic orbifold. As in the parabolic orbifold, this can be understood either as a result of the singularity of the vertex operators at $x^+ = 0$, or as a consequence of the fact that they have arbitrarily large energy.

In conclusion, we found interesting singularities in S-matrix elements, which are "UV enhanced IR divergences." We believe that their most likely interpretation is associated with large back reaction of the geometry. In the parabolic orbifold (and in the null-brane without sufficient number of noncompact dimensions) they signal breakdown of perturbation theory. A similar problem exists in the null-brane (and in flat space) for vertex operators with fixed J. However, the S-matrix of the wave packets U of (3.12) in the null-brane with some noncompact dimensions appears to be consistent. Thus these models provide good laboratories for studying aspects of string perturbation theory in a time-dependent setting and should be investigated more thoroughly.

7. Some general comments on the class of free-field time-dependent orbifolds

7.1 Criteria for nonsingular physics

In this section we make some preliminary remarks on the classification of those time-dependent string orbifold models to which the the methods of [15] can be straightforwardly applied. We consider spacetimes of the form $(\mathbb{R}^{1,d+1}/\Gamma) \times \mathcal{C}^{\perp}$ where $\Gamma \subset \mathcal{P}(1, d+1)$ is a discrete subgroup of the Poincaré group $\mathcal{P}(1, d+1)$ in $\mathbb{R}^{1,d+1}$ and $\mathcal{C}^{\perp}$ stands for some transverse spacetime (more precisely, some transverse conformal field theory) on which Γ does not act.

In order to state any classification result one must formulate carefully the list of desired properties. We would like to have:

1. Time-dependence
2. Time-orientability
3. Some unbroken supersymmetry
4. No CTC's

   Unbroken supersymmetry and time-dependence implies that there is a null Killing vector and hence spacetimes satisfying these criteria are foliated by hypersurfaces $\mathcal{F}_{x^+}$ of constant x^+, where ∂_{x^-} is null [3]. A further criterion one might wish to impose is:

5. For generic x^+ there are no closed null curves (CNC's).

   Somewhat surprisingly, these criteria (and even the stronger condition that for *all* x^+ there are no closed causal curves) do not guarantee nonsingularity of the physics. Things can go wrong if there is a family of homotopically inequivalent spacelike curves γ_n whose lengths $L(\gamma_n)$ approach zero for $n \to \infty$. We refer to such a family as *nearly closed null curves* (NCNC's). To eliminate this pathology we might also wish to impose:

6. There is a positive constant κ such that all closed homotopically nontrivial curves have length $L(\gamma) > \kappa$.

7.2 Classification result for $d = 2$

In this case one can identify spacetime with 2×2 hermitian matrices

$$\mathbf{X} = \begin{pmatrix} \sqrt{2}x^- & x \\ \overline{x} & \sqrt{2}x^+ \end{pmatrix} \tag{7.1}$$

where $x \in C$, $x^\pm \in \mathbb{R}$. The Minkowski metric is

$$ds^2 = -\det(d\mathbf{X}) \tag{7.2}$$

and the action of the Poincaré group is

$$\mathbf{X} \to g\mathbf{X}g^\dagger + \mathbf{A} \tag{7.3}$$

where $g \in \mathrm{SL}(2, \mathbb{C})$. We denote elements of the Poincaré group by $\Lambda = (g, \mathbf{A})$.

We now classify the possible orbifold groups Γ. It is not difficult to show that any discrete group satisfying the criteria 1 – 4 above can be conjugated into a subgroup of the continuous group $\mathcal{S} \subset \mathcal{P}(1, 3)$ defined by elements of the form

$$\Lambda = \left(g = \begin{pmatrix} 1 & \xi/\sqrt{2} \\ 0 & 1 \end{pmatrix}; \mathbf{A} = \begin{pmatrix} \sqrt{2}a^- & a \\ \overline{a} & 0 \end{pmatrix} \right) \qquad \xi, a \in \mathbb{C}, a^- \in \mathbb{R} \tag{7.4}$$

Every element of $\mathcal{S}$ can be written as $t(a^-)\Lambda(\xi, a)$ with

$$\Lambda(\xi, a) := \left(g = \begin{pmatrix} 1 & \xi/\sqrt{2} \\ 0 & 1 \end{pmatrix}; \mathbf{A} = \begin{pmatrix} 0 & a \\ \overline{a} & 0 \end{pmatrix} \right) \qquad \xi, a \in \mathbb{C} \tag{7.5}$$

and

$$t(a^-) := \left(g = 1; \mathbf{A} = \begin{pmatrix} \sqrt{2}a^- & 0 \\ 0 & 0 \end{pmatrix} \right) \qquad a^- \in \mathbb{R}. \tag{7.6}$$

From the identities:

$$(a^-)\Lambda(\xi, a) = \Lambda(\xi, a)t(a^-) \tag{7.7}$$

$$\Lambda(\xi_1, a_1)\Lambda(\xi_2, a_2) = t(\mathrm{Re}(\xi_1\overline{a}_2))\Lambda(\xi_1 + \xi_2, a_1 + a_2) \tag{7.8}$$

it follows that, group-theoretically, $\mathcal{S}$ is the five-dimensional Heisenberg group.

JHEP10(2002)031

If we also impose criterion 5 above then Γ must not contain any group elements of the form $t(a^-)$ and moreover must be an abelian subgroup of $\mathcal{S}$. This will be true if $\mathrm{Re}(\xi_1\overline{a}_2 - \xi_2\overline{a}_1) = 0$ for all pairs of elements.

We may now organize the classification by the minimal number r of generators of Γ.

If $r = 1$, then, using conjugation by elements of $\mathcal{P}(1,3)$ it is easy to show that the most general possibility is the null-brane, together with the parabolic orbifold.[7] That is, we can take $g_1 = \Lambda(2\pi, iR)$ for $R \in \mathbb{R}$. When $r \geq 2$ we take orbifolds of the null brane. When $r = 2$ the inequivalent possibilities for the second generator $g_2 = t(a_2^-)\Lambda(\xi_2, a_2)$ fall into two cases depending on whether or not we can conjugate a_2^- to zero:[8]

1. $(a_2^- \in \mathbb{R}^*, \xi_2 \in \mathbb{R}, 2\pi a_2 = iR\xi_2)$
2. $(a_2^- = 0, \xi_2, a_2 \in \mathbb{C})$.

The two-generator model with generators $\Lambda(\xi_i, a_i)$, $i = 1,2$, has an interesting geometry exhibiting the NCNC's mentioned above. The action of the group on x is given by $x \to x + n_i\omega_i$ with $\omega_i = \xi_i x^+ + a_i$. Thus the leaves of the foliation $\mathcal{F}_{x^+}$ are themselves foliated by tori with time-dependent periods ω_i. If ω_1, ω_2 are not rationally related there are no CNC's and no fixed points. Nevertheless, the physics is potentially singular. The reason is that at (generically two) critical times x_c^+ the tori degenerate, and $\omega_2(x^+) = \lambda\omega_1(x^+)$ for a real number λ. When λ is rational there are CNC's. If λ is irrational then by Dirichlet's theorem there is an infinite set of distinct solutions in relatively prime integers (p,q) to $|p\lambda - q| < 1/|q|$ and the geodesics that lift to $q\omega_1 - p\omega_2$ on the covering space form an infinite set of closed spacelike geodesics, γ_n, in nontrivial homotopy classes, whose lengths $L(\gamma_n)$ approach zero for $n \to \infty$. Then, for example, the classical soliton sum (with $\alpha' = 1$)

$$\sum_{n_i^a, n_i^b \in Z} \exp\left\{-\frac{\pi}{\mathrm{Im}\,\tau}\left|\left(n_1^a - \tau n_1^b\right)\left(\xi_1 x^+ + a_1\right) + \left(n_2^a - \tau n_2^b\right)\left(\xi_2 x^+ + a_2\right)\right|^2\right\} \tag{7.9}$$

diverges. This leads to potential divergences in one-loop amplitudes. Similarly, there are potential divergences in propagators and vev's of composite operators. While such divergences are perhaps cured by supersymmetric cancellations, we will see below that the NCNC's lead to singularities in wavefunctions analogous to those of the parabolic orbifold.

One can show that for $r \geq 3$ generators the generic model has no CNC's or fixed points but suffers from NCNC's. The difference from the $r = 2$ case is that now this happens for all x^+, since for all x^+ there will be infinitely many Diophantine approximations $n_1\omega_1 + n_2\omega_2 + n_3\omega_3 \cong 0$.

In view of these remarks we are motivated to impose criterion 6 above. It then follows from the classification of the two generator models above that:

The only time dependent orbifolds of $\mathbb{R}^{1,3}$ *with nonsingular physics are the null branes with* $R > 0$.

[7] This case is very closely related to the discussion of Figueroa-O'Farrill and Simón [4]. In our language, they classified 1-generator models for $\mathbb{R}^{1,10}$. Although their assumptions were more restrictive than ours, the outcome is the same, at least, for $\mathbb{R}^{1,3}$.

[8] At this point we have not yet imposed the condition that Γ is abelian.

7.3 Comments on the orbifold groups for $d > 2$

It would be very interesting to extend the analysis of the previous section to $9+1$ dimensions. It is possibly useful in this context to represent lorentzian spacetime as 2×2 hermitian matrices over the octonions $\mathbf{O}$ since one can then identify the Lorentz group with $\mathrm{SL}(2, \mathbf{O})$ (suitably-defined) [40]. However we will follow another approach using the results of [3]. We will only make some general comments and will not achieve a full classification of the possibilities.

Using the results of appendix A of [3] is straightforward to show that the criterion of some unbroken supersymmetry and time dependence implies that the orbifold group $\Gamma \subset \mathcal{P}(1,9)$ must be a discrete subgroup of

$$G = (\mathrm{Spin}(7) \ltimes \mathbb{R}^8) \ltimes \mathbb{R}^{1,9} \tag{7.10}$$

To describe the action of G as a group of Poincaré transformations we denote vectors by $(x^+, \vec{x}, x^-) \in \mathbb{R}^{1,9}$ with $\vec{x} \in \mathbb{R}^8$. The general group element then acts by

$$\begin{aligned} x^+ &\to x^+ + R^+ \\ \vec{x} &\to \sigma \cdot (\vec{x} + \vec{v} x^+ + \vec{R}) \\ x^- &\to x^- + \vec{v} \cdot \vec{x} + \frac{1}{2} \vec{v}^2 x^+ + R^- \end{aligned} \tag{7.11}$$

where $R^\pm$ are real, $\vec{v}, \vec{R} \in \mathbb{R}^8$, and σ is a rotation in the Spin(7) subgroup of the $SO(8)$ isometry group of $\mathbb{R}^8$ fixing the covariantly constant spinor.

A natural subgroup to consider is that generated by the transformations (7.11) with $R^+ = 0$. A simple computation of $(x - \Lambda \cdot x)^2$ shows that identifications by elements of this subgroups do not generate CTC's, and in this sense the subgroup defined by $R^+ = 0$ may be considered an analog of the group $\mathcal{S}$ of section 7.2. It is generated by Spin(7) rotations together with translations in x^- by R^-, denoted $t(R^-)$, and the transformations $\Lambda(\vec{v}, \vec{R})$ defined by (7.11) with $R^+ = R^- = 0$ and $\sigma = 1$. The subgroup of elements $t(R^-)\Lambda(\vec{v}, \vec{R})$ which is $\mathbb{R}^8 \ltimes (\mathbb{R}^8 \oplus \mathbb{R})$ forms a 17-dimensional Heisenberg group generalizing the 5-dimensional Heisenberg group we found before. Indeed: $t(R^-)$ is central in G and

$$\Lambda\left(\vec{v}_1, \vec{R}_1\right) \Lambda\left(\vec{v}_2, \vec{R}_2\right) = t\left(\vec{v}_1 \cdot \vec{R}_2\right) \Lambda\left(\vec{v}_1 + \vec{v}_2, \vec{R}_1 + \vec{R}_2\right) . \tag{7.12}$$

There is an analogous Heisenberg group acting on $\mathbb{R}^{1,d+1}$ for any d.

The presence of the Spin(7) factor in G is a new feature of higher dimensions, not seen in the 3+1 dimensional null brane discussed above. The inclusion of rotations in the classification of models is a very interesting subject to which we hope to return. One important point is that the rotations in Spin(7) cannot be arbitrary, even when accompanied by a shift $\vec{R}$. This is implied by the convergence of the 1-loop amplitude. The basic point can already be made in the standard Melvin model $(\mathbb{C} \times \mathbb{R})/\mathbb{Z}$, defined by the orbifold action $(z, y) \to (e^{2\pi i \gamma} z, y + R)$ where z is complex, y is real, and γ is a real rotation angle. The convergence of the 1-loop partition function for a fixed τ is controlled by the convergence of

$$\sum_{s \in \mathbb{Z}} e^{-\beta s^2} \frac{1}{\parallel s\gamma \parallel^2} \tag{7.13}$$

where $\| s\gamma \|$ is the distance to the nearest integer. The denominator in (7.13) arises from the entropy of states in the s twisted sector. If this entropy grows with s sufficiently fast, the sum (7.13) can diverge. In number theory it is shown that there are some irrational numbers γ such that there can be a series of extremely good rational approximations to γ. For these γ's the denominator term in (7.13) can overwhelm the gaussian falloff term in (7.13). An example of such an irrational number γ can be constructed as follows. Let $f(n)$ be a very rapidly growing function, for example $f(n+1) = A^{f(n)}$ for a large constant A, and set

$$\gamma = \sum_{n=1}^{\infty} \frac{1}{10^{f(n)}} \tag{7.14}$$

Then, equation (7.13) diverges. In the remainder of this paper we consider only quotients by subgroups of the Heisenberg group.

Turning our attention now to the causal structure of the quotient spaces $\mathcal{F}_{x^+} = \mathbb{R}^{d+1}/\Gamma$ we note that there will be no NCNC's if the generating vectors $\vec{v}_i x^+ + \vec{R}_i$ of the torus in $\mathbb{R}^d$ are linearly independent (over $\mathbb{R}$) for all x^+. Put differently, if we assemble these vectors into an $d \times r$ matrix $vx^+ + R$, we require that this matrix have constant rank for all x^+.

7.4 Wavefunctions on the generalized orbifolds

Now let us discuss the analog of the U-functions for general orbifolds associated with subgroups of the Heisenberg group discussed above. The general solution of the wave equation with fixed $p^+ \neq 0$ in $\mathbb{R}^{1,d+1}$ can be written as:

$$U_{\chi,p^+}(X) := \int_{\mathbb{R}^d} d\vec{p}\, \chi(\vec{p})\, e^{i(P,X)} \,. \tag{7.15}$$

Here we write $X = (x^+, \vec{x}, x^-)$ with $\vec{x} \in \mathbb{R}^d$ and similarly

$$P = (p^+, \vec{p}, p^- = \frac{\vec{p}^2 + m^2}{2p^+}) \,, \tag{7.16}$$

while $(P,X) = -p^+x^- - p^-x^+ + \vec{p}\cdot\vec{x}$ is the Lorentz invariant inner product. It is easy to show that

$$U_{\chi,p^+}(\Lambda(\vec{v},\vec{R})\cdot X) = \int_{\mathbb{R}^d} d\vec{p}\, \chi(\vec{p}+\vec{v}p^+)\; e^{i(\vec{p}+p^+\vec{v})\cdot\vec{R}}\; e^{i(P,X)} \tag{7.17}$$

Now let us consider wavefunctions on an orbifold with r generators $g_i = \Lambda(\vec{v}_i, \vec{R}_i), i = 1, \ldots, r$. From (7.17) we learn that invariant wavefunctions must satisfy

$$\chi(\vec{p}+\vec{v}_i p^+) = e^{-i(\vec{p}+p^+\vec{v}_i)\cdot\vec{R}_i}\chi(\vec{p}) \qquad i = 1, \ldots, r \tag{7.18}$$

Let us now consider the two sources of difficulty, identified below equation (3.11), in the amplitudes. First we examine the condition of nonsingularity of the wavefunction for all x^+. In order to analyze it, we need to find a more explicit solution to (7.18). Let $H_v \subset \mathbb{R}^d$ be the linear span of the vectors $\vec{v}_i$. Suppose we can find a quadratic form Q on $\mathbb{R}^d$ and a vector $\vec{b} \in \mathbb{R}^d$ such that

$$\begin{aligned} p^+ Q\vec{v}_i &= \vec{R}_i \qquad i = 1, \ldots r \\ \vec{b}\cdot\vec{v}_i &= \frac{1}{2}\vec{v}_i\cdot\vec{R}_i \qquad i = 1, \ldots, r \,. \end{aligned} \tag{7.19}$$

Then a solution to equation (7.18) can be written as

$$\chi(\vec{p}) = e^{-i\frac{1}{2}\vec{p}Q\vec{p}-i\vec{b}\cdot\vec{p}}\widetilde{\chi}(\vec{p}) \tag{7.20}$$

with

$$\widetilde{\chi}(\vec{p}+p^+\vec{v}) = \widetilde{\chi}(\vec{p}) \tag{7.21}$$

i.e. $\widetilde{\chi}$ is a periodic function along the hyperplane H_v spanned by $\vec{v}_i$ with periods $p^+\vec{v}_i$, and can be written as a sum of plane waves.

At this point, we sacrifice some generality and assume that $\vec{v}_i$ are linearly independent and therefore H_v has dimension r. Then it is easy to find $Q, \vec{b}$ as follows. Construct $\vec{w}_i, i = 1, \ldots, r$ and $\vec{u}_{i'}, i' = 1, \ldots d-r$ such that

$$\begin{aligned} w_{aj}v_{ai} &= \delta_{ij} \\ u_{aj'}v_{ai} &= 0 \\ u_{ai'}u_{aj'} &= \delta_{i'j'}\,. \end{aligned} \tag{7.22}$$

Here and below, repeated indices are summed. We can now decompose $\vec{R}_i$ in the basis $\vec{w}_i, \vec{u}_{i'}$:

$$R_{ai} = w_{aj}I_{ji} + u_{ai'}N_{i'i}\,. \tag{7.23}$$

Note that, by the condition that Γ be abelian, $I_{ji} = \vec{v}_j \cdot \vec{R}_i$ is *symmetric* so we can set

$$p^+Q_{ab} = w_{ai}I_{ij}w_{bj} + u_{ai'}N_{i'i}w_{bi} + u_{bi'}N_{i'i}w_{ai} \tag{7.24}$$

while $\vec{b} = \sum_{i=1}^r \frac{1}{2}(\vec{v}_i \cdot \vec{R}_i)\vec{w}_i$ will do.

Now let us ask if the wavefunctions U_{χ,p^+} become singular for any x^+. We introduce coordinates in momentum space appropriate to the decomposition $\mathbb{R}^d = H_v \oplus H_v^\perp$:

$$p_a = v_{ai}p_i + u_{ai'}q_{i'} \tag{7.25}$$

Then from (7.21) we can Fourier decompose and write:

$$\widetilde{\chi}(p_i, q_{i'}) = \sum_{\vec{n}} \widetilde{\chi}_{\vec{n}}(q_{i'})e^{2\pi i n_i p_i/p^+} \tag{7.26}$$

and $\widetilde{\chi}_{\vec{n}}(q_{i'})$ can be functions of rapid decrease. Let us similarly write

$$X_a = w_{ai}y_i + u_{ai'}z_{i'} \tag{7.27}$$

Then, up to a constant, the wavefunction U_{χ,p^+} can be written:

$$\begin{aligned} &e^{-ip^+x^-}\sum_{\vec{n}}\int\prod_{i'=1}^{d-r}dq_{i'}\widetilde{\chi}_{\vec{n}}(q_{i'})e^{-\frac{i}{2p^+}(q_{i'}q_{i'}+m^2)}e^{iz_{i'}q_{i'}}\times \\ &\times\int\prod_{i=1}^{r}dp_i e^{-\frac{i}{2p^+}p_ip_j(I_{ij}+x^+v_i\cdot v_j)}e^{-\frac{i}{p^+}q_{i'}N_{i'i}p_i}e^{ip_i\left(y_i-\frac{1}{2}(\vec{v}_i\cdot\vec{R}_i)+2\pi n_i/p^+\right)}\,. \end{aligned} \tag{7.28}$$

Now we see that the condition of nonsingularity of the wavefunction for all x^+ fits in very nicely with the condition for the absence of NCNC's. Recall that the condition for the absence of NCNC's is that the $d \times r$ matrix

$$v_{ai} x^+ + R_{ai} \tag{7.29}$$

have constant rank for all x^+. Suppose this condition is satisfied. Then, there is no possible singularity in U_{χ,p^+} unless the quadratic form $I_{ij} + x^+ \vec{v}_i \cdot \vec{v}_j$ has a nonzero eigenvector p_i^* of eigenvalue zero. By our assumption on the rank of v_{ai} this will only happen at special critical times x_c^+. In this case the gaussian integral on p_i degenerates to a delta function. However, as long as this vector p_i^* is not also in the kernel of $N_{i'i}$ the argument of the delta function depends nontrivially on the q_i' and the subsequent $q_{i'}$ integral is nonsingular. But the condition that any null vector of $I_{ij} + x^+ \vec{v}_i \cdot \vec{v}_j$ is not in the kernel of $N_{i'i}$ is precisely the condition that (7.29) have constant rank!

Thus, we conclude that the condition of the nonsingularity of the wavefunctions is precisely the condition for the absence of nearly closed null curves in the quotient $\mathcal{F}_{x^+} = \mathbb{R}^{d+1}/\Gamma$ *for all* x^+.

Let us finally comment briefly on the suppression of the large energy component. Evidently, since χ in (7.15) is a (quasi) periodic function along H_v, it cannot have rapid decrease in these directions. It can have rapid decrease along the orthogonal space $H_v^\perp$. For example, in (7.26) we can take $\widetilde{\chi}_{\vec{n}}(q_{i'})$ to be functions of $q_{i'}$ of rapid decrease. Then, as in our discussion in the null brane case in section 3, it is possible to suppress large momenta along H_v provided that $N_{i'i}$ satisfies certain properties. More explicitly, from (7.28) the integration over $q_{i'}$ with $\widetilde{\chi}_{\vec{n}}(q_{i'})$ will generate a rapidly decreasing function of $\sum_i N_{i'i} p_i$ for fixed z_i. Thus if $\sum_i N_{i'i} p_i$ span[9] H_v then large momenta in all directions including H_v will be suppressed.

Acknowledgments

We thank S. Sethi for stressing to us the importance of the null-brane. We also thank G. Horowitz, J. Polchinski and E. Silverstein for encouraging us to examine the null-brane. We thank C. Bachas, T. Banks, J. Froehlich, G. Horowitz, C. Hull, J. Maldacena, J. Polchinski, S. Shenker, E. Silverstein, L. Susskind, and E. Witten for discussions. H.L. and G.M. were supported in part by DOE grant #DE-FG02-96ER40949 to Rutgers. N.S. was supported in part by DOE grant #DE-FG02-90ER40542 to IAS. GM would like to thank L. Baulieu and B. Pioline for hospitality at the LPTHE and the Isaac Newton Institute for hospitality during the completion of this work.

References

[1] G.T. Horowitz and A.R. Steif, *Singular string solutions with nonsingular initial data*, *Phys. Lett.* **B 258** (1991) 91.

[2] G.T. Horowitz and A.R. Steif, *Strings in strong gravitational fields*, *Phys. Rev.* **D 42** (1990) 1950.

[9] This requires that $r \leq d - r$ and the matrix $N_{i'i}$ should have rank r.

[3] J.M. Figueroa-O'Farrill, *Breaking the m-waves*, *Class. and Quant. Grav.* **17** (2000) 2925 [hep-th/9904124].

[4] J. Figueroa-O'Farrill and J. Simon, *Generalized supersymmetric fluxbranes*, *J. High Energy Phys.* **12** (2001) 011 [hep-th/0110170].

[5] J. Khoury, B.A. Ovrut, N. Seiberg, P.J. Steinhardt and N. Turok, *From big crunch to big bang*, *Phys. Rev.* **D 65** (2002) 086007 [hep-th/0108187];
N. Seiberg, *From big crunch to big bang-is it possible?*, hep-th/0201039.

[6] V. Balasubramanian, S.F. Hassan, E. Keski-Vakkuri and A. Naqvi, *A space-time orbifold: a toy model for a cosmological singularity*, hep-th/0202187.

[7] L. Cornalba and M.S. Costa, *A new cosmological scenario in string theory*, *Phys. Rev.* **D 66** (2002) 066001 [hep-th/0203031].

[8] N.A. Nekrasov, *Milne universe, tachyons and quantum group*, hep-th/0203112.

[9] E. Kiritsis and B. Pioline, *Strings in homogeneous gravitational waves and null holography*, *J. High Energy Phys.* **08** (2002) 048 [hep-th/0204004].

[10] A.J. Tolley and N. Turok, *Quantum fields in a big crunch/big bang spacetime*, hep-th/0204091.

[11] J. Simon, *The geometry of null rotation identifications*, *J. High Energy Phys.* **06** (2002) 001 [hep-th/0203201].

[12] M. Gutperle and A. Strominger, *Spacelike branes*, *J. High Energy Phys.* **04** (2002) 018 [hep-th/0202210].

[13] A. Sen, *Rolling tachyon*, *J. High Energy Phys.* **04** (2002) 048 [hep-th/0203211].

[14] O. Aharony, M. Fabinger, G.T. Horowitz and E. Silverstein, *Clean time-dependent string backgrounds from bubble baths*, *J. High Energy Phys.* **07** (2002) 007 [hep-th/0204158].

[15] H. Liu, G. Moore and N. Seiberg, *Strings in a time-dependent orbifold*, *J. High Energy Phys.* **06** (2002) 045 [hep-th/0204168].

[16] S. Elitzur, A. Giveon, D. Kutasov and E. Rabinovici, *From big bang to big crunch and beyond*, *J. High Energy Phys.* **06** (2002) 017 [hep-th/0204189].

[17] L. Cornalba, M.S. Costa and C. Kounnas, *A resolution of the cosmological singularity with orientifolds*, *Nucl. Phys.* **B 637** (2002) 378 [hep-th/0204261].

[18] B. Craps, D. Kutasov and G. Rajesh, *String propagation in the presence of cosmological singularities*, *J. High Energy Phys.* **06** (2002) 053 [hep-th/0205101].

[19] V. Balasubramanian and S.F. Ross, *The dual of nothing*, *Phys. Rev.* **D 66** (2002) 086002 [hep-th/0205290].

[20] S. Kachru and L. McAllister, *Bouncing brane cosmologies from warped string compactifications*, hep-th/0205209.

[21] A. Lawrence, *On the instability of 3d null singularities*, hep-th/0205288.

[22] G. Horowitz and J. Polchinski, to appear.

[23] G.W. Gibbons, *Quantized fields propagating in plane wave space-times*, *Commun. Math. Phys.* **45** (1975) 191.

[24] A.A. Tseytlin, *Exact string solutions and duality*, hep-th/9407099;
C. Klimčík and A.A. Tseytlin, unpublished 1994;
A.A. Tseytlin, unpublished 2001.

[25] D. Robbins and S. Sethi, to appear.

[26] M. Fabinger and J. McGreevy, to appear.

[27] M. Blau, J. Figueroa-O'Farrill and G. Papadopoulos, *Penrose limits, supergravity and brane dynamics*, *Class. and Quant. Grav.* **19** (2002) 4753 [hep-th/0202111].

[28] M. Bañados, C. Teitelboim and J. Zanelli, *The black hole in three-dimensional space-time*, *Phys. Rev. Lett.* **69** (1992) 1849 [hep-th/9204099].

[29] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, *Geometry of the (2+1) black hole*, *Phys. Rev.* **D 48** (1993) 1506 [gr-qc/9302012].

[30] J.G. Russo and A.A. Tseytlin, *Magnetic flux tube models in superstring theory*, *Nucl. Phys.* **B 461** (1996) 131 [hep-th/9508068].

[31] T. Takayanagi and T. Uesugi, *Orbifolds as Melvin geometry*, *J. High Energy Phys.* **12** (2001) 004 [hep-th/0110099].

[32] T. Takayanagi and T. Uesugi, *D-branes in Melvin background*, *J. High Energy Phys.* **11** (2001) 036 [hep-th/0110200].

[33] M. Reed and B. Simon, *Methods of mathematical physics vol. 3: scattering theory*, Academic, New York 1979.

[34] H. Liu, G. Moore, and N. Seiberg, to appear.

[35] R. Rohm, *Spontaneous supersymmetry breaking in supersymmetric string theories*, *Nucl. Phys.* **B 237** (1984) 553.

[36] I.J. Muzinich and M. Soldate, *High-energy unitarity of gravitation and strings*, *Phys. Rev.* **D 37** (1988) 359.

[37] D. Amati, M. Ciafaloni and G. Veneziano, *Superstring collisions at planckian energies*, *Phys. Lett.* **B 197** (1987) 81;
D. Amati, M. Ciafaloni and G. Veneziano, *Classical and quantum gravity effects from planckian energy superstring collisions*, *Int. J. Mod. Phys.* **A 3** (1988) 1615.

[38] T. Banks, *Cosmological breaking of supersymmetry or little lambda goes back to the future. ii*, hep-th/0007146.

[39] E. Witten, *Quantum gravity in de Sitter space*, hep-th/0106109.

[40] C.A. Manogue and J. Schray, *Finite Lorentz transformations, automorphisms and division algebras*, *J. Math. Phys.* **34** (1993) 3746 [hep-th/9302044].

JHEP10(2002)031